

## How-To: Transformations with Matrices

Say you have a triangle with coordinates: A(2, 3), B(-3, 5) and C(0, 1).

The coordinates of the triangle  $(x_1, y_1), (x_2, y_2), (x_3, y_3)$  can be written as a matrix of the form  $\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$

So the VERTEX matrix for our triangle is  $\begin{bmatrix} 2 & -3 & 0 \\ 3 & 5 & 1 \end{bmatrix}$

We can transform this triangle like we did in Geometry.

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### 1. Translations (moving the shape around)

EX: Suppose the triangle is shifted 4 units up and 3 units to the left.

To do this, you would subtract 3 from each x-coordinate and add 4 to each y-coordinate.

We can replicate this by creating a TRANSLATION matrix which has the same dimensions as the vertex matrix:

$$\begin{bmatrix} -3 & -3 & -3 \\ 4 & 4 & 4 \end{bmatrix}$$

To get the coordinates of the new triangle, ADD the translation matrix to the coordinate matrix:

$$\begin{bmatrix} 2 & -3 & 0 \\ 3 & 5 & 1 \end{bmatrix} + \begin{bmatrix} -3 & -3 & -3 \\ 4 & 4 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -6 & -3 \\ 7 & 9 & 5 \end{bmatrix} \rightarrow \text{the new coordinates are } A'(-1, 7), B'(-6, 9) \text{ and } C'(-3, 5)$$

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### 2. Dilations (increasing or decreasing the size)

EX: Suppose the triangle is doubled in size.

To do this, you multiply the coordinate matrix by a factor of 2. This is scalar multiplication.

$$2 \cdot \begin{bmatrix} 2 & -3 & 0 \\ 3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -6 & 0 \\ 6 & 10 & 2 \end{bmatrix} \rightarrow \text{the new coordinates are } A'(4, 6), B'(-6, 10), \text{ and } C'(0, 2)$$

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Note: the matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  is called the IDENTITY matrix.

Multiplying a matrix by the identity matrix results in the original matrix:  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 & 0 \\ 3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 \\ 3 & 5 & 1 \end{bmatrix}$

The identity matrix maps the x and y values like this:

*the new x = 1 · (the old x) and the new y = 1 · (the old y)*

So the matrix  $\begin{matrix} \text{new } x - & \text{old } x & \text{old } y \\ \text{new } y - & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{matrix}$  does this.

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### 3. Reflections (the mirror image over a line)

EX: Suppose the triangle is reflected over the y-axis.

If a random point, for example (2, 3) is reflected over the y-axis, it becomes (-2, 3).

Therefore:  $the\ new\ x = -1 \cdot (the\ old\ x)$  and  $the\ new\ y = 1 \cdot (the\ old\ y)$

This requires a REFLECTION matrix:  $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$   $\begin{matrix} new\ x = \\ new\ y = \end{matrix}$   $\begin{matrix} old\ x & old\ y \\ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \end{matrix}$

To find the coordinates of our triangle reflected over the y-axis, multiply the reflection matrix by the vertex matrix:  $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 & 0 \\ 3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 0 \\ 3 & 5 & 1 \end{bmatrix}$

Observing what happens with other reflections, we can create the following reflection matrices:

Reflection over the x-axis:  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  Reflection over the line  $y = x$ :  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

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### 4. Rotations:

Rotations also have their own ROTATION matrices:

$90^\circ\ ccw = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$   $180^\circ\ ccw = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$   $270^\circ\ ccw = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

EX: If the triangle was rotated  $270^\circ$ , the new coordinates would be:

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 & 0 \\ 3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 1 \\ -2 & 3 & 0 \end{bmatrix}$  or  $A'(3, -2)$ ,  $B'(5, 3)$  and  $C'(1, 0)$